

On the penultimate tail behavior of Weibull-type models

Abstract: The Gumbel max-domain of attraction corresponds to a null tail index which do not distinguish the different tail weights that might exist between distributions within this class. The Weibull-type distributions form an important subgroup of this latter and includes the so-called *Weibull-tail coefficient*, usually denoted θ , that specifies the tail behavior, with larger values indicating slower tail decay. Here we shall see that the Weibull-type distributions present a penultimate tail behavior Fréchet if $\theta > 1$ and a penultimate tail behavior Weibull whenever $\theta < 1$.

Keywords: Extreme value theory, penultimate distributions, Weibull-type models

1 Introduction

The main objective of an extreme value analysis is to estimate the probability of events that are more extreme than any that have already been observed. By way of example, suppose that a sea-wall projection requires a coastal defense from all sea-levels, for the next 100 years. The use of extremal models enables extrapolations of this type. The central result in classical Extreme Value Theory (EVT) states that, for an i.i.d. sequence, $\{X_n\}_{n \geq 1}$, having common distribution function (d.f.) F , if there are real constants $a_n > 0$ and b_n such that,

$$P(\max(X_1, \dots, X_n) \leq a_n x + b_n) = F^n(a_n x + b_n) \xrightarrow{n \rightarrow \infty} G_\gamma(x), \quad (1)$$

for some non degenerate function G_γ , then it must be the Generalized Extreme Value function (GEV),

$$G_\gamma(x) = \exp(-(1 + \gamma x)^{-1/\gamma}), \quad 1 + \gamma x > 0, \quad \gamma \in \mathbb{R},$$

($G_0(x) = \exp(-e^{-x})$) and we say that F belongs to the max-domain of attraction of G_γ , in short, $F \in \mathcal{D}(G_\gamma)$. The parameter γ , known as the tail index, is a shape parameter as it determines the tail behavior of F , being so a crucial issue in EVT. More precisely, if $\gamma > 0$ we are in the domain of attraction Fréchet corresponding to a heavy tail, $\gamma < 0$ indicates the Weibull domain of attraction of light tails and $\gamma = 0$ means a Gumbel domain of attraction and an exponential tail.

However, as Fisher and Tippett ([4], 1928) remarked, if one approximates the distribution of the successive maxima of normal samples not by the limit distribution Gumbel but by a sequence of other extreme value distributions converging to the limit distribution, the approximation is asymptotically improved. They called penultimate distributions to this sequence of approximating extreme value distributions.

Here we will analyze the penultimate tail behavior of the Weibull-type models which are a wide class in the Gumbel max-domain.

Weibull-type models have representation

$$1 - F(x) = \exp(-H(x)), \quad x \geq x_0 \geq 0, \quad \theta > 0 \quad (2)$$

where

$$H(x) = x^{1/\theta} l(x) \text{ or } H^{-1}(x) = x^\theta l^*(x), \quad (3)$$

with H^{-1} denoting the generalized inverse of H and functions l and l^* are slowly varying at infinity (i.e., $l(tx)/l(t) \rightarrow 1$ as $t \rightarrow \infty$ for all $x > 0$ and the same holds for l^*).

We say that functions H and H^{-1} are regularly varying with indexes, $1/\theta$ and θ , respectively.

The parameter θ , called the Weibull-tail coefficient, governs the tail behavior of F , with larger values indicating slower tail decay. The Weibull-type distributions form an important subgroup

within the Gumbel class ($\gamma = 0$) and the tail behavior can then be specified using the Weibull-tail coefficient (Dierckx *et al.*, [3] 2009).

Weibull-type models include well-known distributions such as, Normal ($\theta = 1/2$), Weibull(α, λ) ($\theta = 1/\alpha$), Extended Weibull(β, δ) ($\theta = 1/\beta$), Exponential, Gamma and Logistic ($\theta = 1$) (Gardes and Girard [5], 2008).

We will prove that the Weibull-type models present penultimate tail behavior Fréchet or Weibull whenever $\theta > 1$ or $\theta < 1$, respectively. To this end, we will use a result in de Haan and Gomes ([8], 1999) and in Gomes ([6], 1984).

We remark that the case $\theta = 1$ cannot be treated in a unified way since we can find penultimate tail indexes with different orders. For instance, the Exponential and Logistic have γ_n of order $1/n$ but the Gamma distribution have γ_n of order $1/\log^2 n$ (Gomes [7], 1993). Therefore, we shall assume that $\theta \neq 1$ all over the paper.

2 Results

Consider

$$k(x) := \frac{d}{dx}[-\log(-\log F(x))] \quad (4)$$

Observe that

$$-\log F(x) = (1 - F(x))(1 + O(1 - F(x)))$$

and, by (2),

$$-\log(-\log F(x)) = H(x) - \log(1 + O(e^{-H(x)})),$$

leading to

$$k(x) = H'(x) \left(1 + \frac{O(e^{-H(x)})}{1 + O(e^{-H(x)})} \right) = H'(x)(1 + o(1)). \quad (5)$$

Therefore

$$\begin{aligned} k'(x) &= H''(x) \left(1 + \frac{O(e^{-H(x)})}{1 + O(e^{-H(x)})} \right) + [H'(x)]^2 \frac{O(e^{-2H(x)}) - O(e^{-H(x)})(1 + O(e^{-H(x)}))}{(1 + O(e^{-H(x)}))^2} \\ &= H''(x)(1 + o(1)). \end{aligned} \quad (6)$$

Analogously we derive

$$k''(x) = H'''(x)(1 + o(1)) \text{ and } k'''(x) = H^{(iv)}(x)(1 + o(1)). \quad (7)$$

Consider $H(x)$ given in (3). We have

$$\begin{aligned} H'(x) &= x^{\theta^{-1}-1} l(x) \left(\theta^{-1} + \frac{x l'(x)}{l(x)} \right) \\ H''(x) &= x^{\theta^{-1}-2} l(x) \left(\theta^{-1}(\theta^{-1} - 1) + 2\theta^{-1} \frac{x l'(x)}{l(x)} + \frac{x^2 l''(x)}{l(x)} \right) \\ H'''(x) &= x^{\theta^{-1}-3} l(x) \left(\theta^{-1}(\theta^{-1} - 1)(\theta^{-1} - 2) + 3\theta^{-1}(\theta^{-1} - 1) \frac{x l'(x)}{l(x)} + 3\theta^{-1} \frac{x^2 l''(x)}{l(x)} + \frac{x^3 l'''(x)}{l(x)} \right) \\ H^{(iv)}(x) &= x^{\theta^{-1}-4} l(x) \left(\theta^{-1}(\theta^{-1} - 1)(\theta^{-1} - 2)(\theta^{-1} - 3) + 4\theta^{-1}(\theta^{-1} - 1)(\theta^{-1} - 2) \frac{x l'(x)}{l(x)} \right. \\ &\quad \left. + 4\theta^{-1}(\theta^{-1} - 1) \frac{x^2 l''(x)}{l(x)} + \theta^{-1} \frac{x^3 l'''(x)}{l(x)} + \frac{x^4 l^{(iv)}(x)}{l(x)} \right). \end{aligned} \quad (8)$$

Assuming that, as $x \rightarrow \infty$,

$$\frac{x l'(x)}{l(x)} \rightarrow 0, \quad \frac{x^2 l''(x)}{l(x)} \rightarrow 0, \quad \frac{x^3 l'''(x)}{l(x)} \rightarrow 0 \quad \text{and} \quad \frac{x^4 l^{(iv)}(x)}{l(x)} \rightarrow 0, \quad (9)$$

and applying (5)-(7), we have that

$$\begin{aligned} xk(x) &= \theta^{-1} x^{\theta^{-1}} l(x) (1 + o(1)) \\ x^2 k'(x) &= xk(x) (\theta^{-1} - 1) \\ x^3 k''(x) &= xk(x) (\theta^{-1} - 2)(\theta^{-1} - 1) \\ x^4 k'''(x) &= xk(x) (\theta^{-1} - 3)(\theta^{-1} - 2)(\theta^{-1} - 1) \end{aligned} \quad (10)$$

Note that, if $l(x)$ is monotone for $x \geq x_0 > 0$, then $x l'(x)/l(x) \rightarrow 0$, as $x \rightarrow \infty$. The other conditions in (9) are also satisfied by the most common models.

Observe that equations (5)-(7) hold for the class of d.f.'s

$$-\log F(x) = \exp(-H(x)) \quad (11)$$

which have been studied in Canto e Castro ([2], 1992). More precisely, it was derived the structure of the remainder $F^n(a_n x + b_n) - G_\gamma(x)$, by showing that F is in class \mathbf{A}_1 of Anderson ([1], 1976), i.e.,

$$\lim_{x \rightarrow \infty} \frac{k''(x)}{k(x)k'(x)} = \lim_{x \rightarrow \infty} \frac{x^3 k''(x)}{xk(x)x^2 k'(x)} = \lim_{x \rightarrow \infty} \frac{(\theta^{-1} - 2)}{xk(x)} = 0. \quad (12)$$

leading to

$$F^n(a_n x + b_n) - G_\gamma(x) = \frac{x^2}{2} \left(\frac{k'(b_n)}{k^2(b_n)} \right) g_\gamma(x) (1 + o(1)), \quad (13)$$

uniformly for x in bounded intervals in the support of G_γ , where $g_\gamma(x) = G'_\gamma(x)$, $a_n = 1/k(b_n)$ and $F(b_n) = \exp(-1/n)$. More details can be seen in Canto e Castro ([2], 1992).

Observe that

$$b_n = H^{-1}(\log n), \quad (14)$$

and the rate of convergence of $F^n(a_n x + b_n)$ to $G_\gamma(x)$ is

$$\frac{k'(b_n)}{k^2(b_n)} = \frac{b_n^2 k'(b_n)}{(b_n k(b_n))^2} \sim \frac{\theta^{-1} - 1}{\theta^{-1}(\log n)} = \frac{1 - \theta}{\log n}. \quad (15)$$

Now consider $x^F := \sup\{x : F(x) < 1\}$ and

$$\varphi(t) = (1/k)'(t) = -k'(t)/(k(t))^2. \quad (16)$$

Based on the von Mises' first order condition,

$$\lim_{t \rightarrow x^F} \varphi(t) = 0, \quad (17)$$

the von Mises' second order condition,

$$\lim_{t \rightarrow x^F} \frac{\varphi'(t)}{k(t)\varphi(t)} = 0, \quad (18)$$

and von Mises' type penultimate condition,

$$\lim_{t \rightarrow x^F} \frac{\varphi''(t)}{k(t)\varphi'(t)} = 0, \quad (19)$$

Theorem 1 in Haan and Gomes ([8], 1999) allow us to derive bounds for $(F^n(a_n x + b_n) - G_{\gamma_n}(x))/\gamma'(\log n)$, with a_n and b_n given above, where

$$\gamma(t) = \varphi(H^{-1}(t)) = -\frac{k'(H^{-1}(t))}{k^2(H^{-1}(t))} \quad (20)$$

and the penultimate tail index is

$$\gamma_n = \gamma(\log n) = -k'(b_n)/k^2(b_n). \quad (21)$$

Proposition 2.1. *Weibull-type models satisfy the von Mises' conditions (17)-(19) whenever conditions in (9) are fulfilled. Moreover, they present penultimate tail behavior Fréchet if $\theta > 1$ and Weibull if $\theta < 1$.*

Dem. First observe that $x^F = +\infty$,

$$\varphi'(t) = -\frac{k''(t)k(t) - 2(k'(t))^2}{(k(t))^3}$$

and

$$\varphi''(t) = 6\frac{k'(t)}{(k(t))^2} \left(\frac{k''(t)}{k(t)} - \left(\frac{k'(t)}{k(t)} \right)^2 \right) - \frac{k'''(t)}{(k(t))^2}.$$

Now we have, successively,

$$\lim_{t \rightarrow x^F} \varphi(t) = \lim_{t \rightarrow x^F} -\frac{k'(t)}{(k(t))^2} = \lim_{t \rightarrow x^F} -\frac{t^2 k'(t)}{(tk(t))^2} = \lim_{t \rightarrow x^F} -\frac{(\theta^{-1} - 1)(1 + o(1))}{\theta^{-1} t^{\theta^{-1}} l(t)(1 + o(1))} = 0, \quad (22)$$

by (12) and (22),

$$\lim_{t \rightarrow x^F} \frac{\varphi'(t)}{k(t)\varphi(t)} = \lim_{t \rightarrow x^F} \left(2\frac{k'(t)}{(k(t))^2} - \frac{k''(t)}{k(t)k'(t)} \right) = 0, \quad (23)$$

and, applying (10),

$$\begin{aligned} \lim_{t \rightarrow x^F} \frac{\varphi''(t)}{k(t)\varphi'(t)} &= \lim_{t \rightarrow x^F} \frac{6k'(t)k''(t)k(t) - 6(k'(t))^3 - k'''(t)(k(t))^2}{(k(t))^2(2(k'(t))^2 - k''(t)k(t))} \\ &= \lim_{t \rightarrow x^F} \frac{6(\theta^{-1} - 1)(\theta^{-1} - 2) - 6(\theta^{-1} - 1)^2 - (\theta^{-1} - 2)(\theta^{-1} - 3)}{tk(t)[2(\theta^{-1} - 1) - (\theta^{-1} - 2)]} = 0. \end{aligned} \quad (24)$$

By (15), the penultimate tail index, γ_n , in (21) is given by

$$\gamma_n = \gamma(\log n) = -k'(b_n)/k^2(b_n) \sim \frac{\theta - 1}{\log n}. \quad (25)$$

Therefore, we obtain $\gamma_n > 0$ and $\gamma_n < 0$ if, respectively, $\theta > 1$ and $\theta < 1$.

Now we compute the rate of convergence $\gamma'(\log n)$. Observe that, after some calculations,

$$\gamma'(t) = \frac{2k'(H^{-1}(t))^2 - k''(H^{-1}(t))k(H^{-1}(t))}{k(H^{-1}(t))^4}, \quad (26)$$

and applying (14) we have

$$\gamma'(\log n) = \frac{2(b_n k(b_n)(\theta^{-1} - 1))^2 - (b_n k(b_n))^2(\theta^{-1} - 1)(\theta^{-1} - 2)}{(b_n k(b_n))^4} \sim \frac{2\theta(1 - \theta)}{(\log n)^2}. \quad \square \quad (27)$$

Remark 2.1. The result in Proposition 2.1 can also be derived based on Gomes ([6], 1984). This latter imposes the condition

$$\lim_{t \rightarrow x^F} \frac{\varphi'(t)}{k(t)(\varphi(t))^2} = c < \infty \quad (28)$$

where $\varphi(t) = (1/k)'(t) = -k'(t)/(k(t))^2$, and obtains boundedness of $(F^n(a_n x + b_n) - G_{\gamma_n}(x))/\gamma_n^2$, with penultimate tail index, γ_n , given in (25).

Observe that

$$\frac{\varphi'(t)}{k(t)(\varphi(t))^2} = 2 - \frac{k''(t)k(t)}{(k'(t))^2}.$$

Applying (10), we obtain

$$\lim_{t \rightarrow x^F} \frac{k''(t)k(t)}{(k'(t))^2} = \lim_{t \rightarrow x^F} \frac{t^3 k''(t) t k(t)}{(t^2 k'(t))^2} = \lim_{t \rightarrow x^F} \frac{(\theta^{-2} - 3\theta^{-1} + 2 + o(1))(1 + o(1))}{(\theta^{-1} - 1 + o(1))^2} = \frac{(\theta^{-1} - 2)(\theta^{-1} - 1)}{(\theta^{-1} - 1)^2}$$

and hence

$$\lim_{t \rightarrow x^F} \frac{\varphi'(t)}{k(t)(\varphi(t))^2} = \frac{1}{1 - \theta}.$$

Thus condition (28) holds for $\theta \neq 1$ (the situation that we are always considering; see the last paragraph of section 1). The rate of convergence is given by $\gamma_n^2 = (\gamma(\log n))^2 = (k'(b_n)/k^2(b_n))^2 = ((\theta - 1)/\log n)^2$, i.e., also the same order obtained in (27).

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